Lecture 13

§7.2. Trigonometric integrals

* Integrals of powers of secant and tangent.

$$\frac{d}{dx}\tan x = \sec^2 x \quad , \quad \sec^2 x = 1+\tan^2 x$$

$$\frac{d}{dx}\sec x = \sec x . \tan x$$

1) If the power of secant is even
use $\sec^2 x = 1+\tan^2 x$, $\frac{d}{dx}\tan x = \sec^2 x$

2) If the power of tangent is odd.
$\underline{\tan^2 x = \sec^2 x - 1}$, $\frac{d}{dx}\sec x = \sec x . \tan x$.

Example: Find $\int \tan^5\theta \sec^7\theta \, d\theta$

Write $I = \int \tan^5\theta \sec^7\theta \, d\theta = \int (\tan^2\theta)^2 \sec^6\theta \, \underline{\tan\theta \sec\theta \, d\theta}$

$$= \int (\sec^2\theta - 1)^2 \sec^6\theta \, \underline{\tan\theta \sec\theta \, d\theta}$$

Put $u = \sec\theta \Rightarrow du = \sec\theta \tan\theta \, d\theta$

$$I = \int (u^2-1)^2 u^6 \, du$$

$$= \int (u^4 - 2u^2 + 1) u^6 \, du$$

$$= \int (u^{10} - 2u^8 + u^6) \, du$$

$$= \frac{u^{11}}{11} - \frac{2u^9}{9} + \frac{u^7}{7} + C.$$

$$= \frac{\sec^{11}\theta}{11} - \frac{2}{9}\sec^9\theta + \frac{1}{7}\sec^7\theta + C.$$

e.g. Use the formula

$\int ta$

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$\theta + \frac{1}{7}\sec^7\theta + C.$

Recall the formula.

$$\int \tan x\,dx = \ln|\sec x| + C.$$

$$* \quad \int \sec x\,dx = \ln|\sec x + \tan x| + C.$$

We have $\displaystyle\int \sec x\,dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$

$$= \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x}\,dx$$

Put $u = \sec x + \tan x \Rightarrow du = (\sec x \tan x + \sec^2 x)dx$

Thus $\displaystyle\int \sec x\,dx = \int \frac{du}{u} = \ln|u| + C$

$$= \ln|\sec x + \tan x| + C$$

Example: Find $\int \tan^3 x\,dx$

$$\int \tan^3 x\,dx = \int \tan^2 x \tan x\,dx$$

$\sec^2 x = 1 + ta$

$$= \int (\sec^2 x - 1)\tan x\,dx$$

$$= \int \tan x \sec^2 x - \int \tan x\,dx$$

$$= \quad I \quad - \quad II \qquad (1)$$

$$I = \int \tan x \sec^2 x\,dx$$

Put $u = \tan x \Rightarrow du = \sec^2 x\,dx$

Thus

$$I = \int u\,du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C \qquad (2)$$

$$II = \int \tan x\,dx = \ln|\sec x| + C' \qquad (3)$$

From (1), (2), (3), we have

$$\int \tan^3 x\,dx = \frac{\tan^2 x}{2} - \ln|\sec x| + C.$$

ple: $\int \sec$ Who can see what you share here 58%

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$\int ... \tan^2 x ... - \ln|\sec x| + C.$

Example: $\int \sec^3 x\, dx$

$\int \sec^2 x \,(\sec x\, dx)$ $\tan x$ $\frac{d(\sec x)}{dx} = \sec x \tan x\, dx$

We cannot use substitution.

↳ Integration by parts

$\begin{cases} u = \sec x \\ dv = \underline{\sec^2 x\, dx} \end{cases} \Rightarrow \begin{cases} du = \sec x \tan x\, dx \\ v = \tan x \end{cases}$

Now $\int \sec^3 x\, dx = uv - \int v\, du$

$= \sec x \tan x - \int \underline{\tan^2 x} \sec x\, dx.$ $\sec^2 ...$ $\to$

$= \sec x \tan x - \int (\sec^2 x - 1)\sec x\, dx$

$\Rightarrow \underline{\int \sec^3 x\, dx} = \sec x \tan x - \underline{\int \sec^3 x\, dx} + \int \sec x\, dx.$

$\Rightarrow 2\int \sec^3 x\, dx = \sec x \tan x + \ln|\sec x + \tan x|$

$\Rightarrow \int \sec^3 x\, dx = \dfrac{\sec x \tan x + \ln|\sec x + \tan x|}{2}$

✱ Some product identities.

a) $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

b) $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

c) $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)].$

Example: $\int \sin 4x \cos 5x\, dx$

$= \frac{1}{2}\int [\sin(5x - 4x) + \sin(5x + 4x)]\, dx$

$= ...$

$= -\frac{1}{2}(\cos x + \cos 9x) ... C$

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* Some product identities.

a) $\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$

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c) $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$.

Example: $\int \sin 4x \cos 5x \, dx$

$$= \frac{1}{2}\int \left[\sin(5x-4x) + \sin(5x+4x)\right] dx$$

$$= \frac{1}{2}\int (\sin x + \sin 9x)\, dx$$

$$= -\frac{1}{2}\left(\cos x + \frac{\cos 9x}{9}\right) + C.$$

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§7.1 Trigonometric integrals

1) $\int \sin^3 x \cos^2 x \, dx$ $\qquad \cos^2 x + \sin^2 x = 1$

$= \int \sin^2 x \cos^2 x \sin x \, dx$ (split)

$= \int (1-\cos^2 x) \cos^2 x \boxed{\sin x \, dx}$

Substitution.

If you see exponent of sin or cos is odd
↳ split it,
⇒ the exponent is even
↳ we can use $\cos^2 x + \sin^2 x = 1$

Put $u = \cos x$, $du = -\sin x \, dx$

$\Rightarrow \int \sin^3 x \cos^2 x \, dx$

$= -\int (1-u^2) u^2 \, du$

$= -\int (u^2 - u^4) \, du$

$= -\dfrac{u^3}{3} + \dfrac{u^5}{5} + C$

$= -\dfrac{\cos^3 x}{3} + \dfrac{\cos^5 x}{5} + C.$

2) $\int \tan x \sec^3 x \, dx$ $\qquad \sec^2 x = 1 + \tan^2 x$

$= \int \tan x (1+\tan^2 x) \sec x \, dx$

If $u = \tan x \Rightarrow du = \sec^2 x \, dx$ → not possible

If $u = \sec x \Rightarrow du = \sec x \tan x \, dx \Rightarrow$ possible

thus $\int \tan x \sec^3 x \, dx$

$= \int \sec^2 x \sec x \tan x \, dx$

Put $u = \sec x \Rightarrow du = \sec x \tan x \, dx$

hence $\int \tan x \sec^3 x \, dx = \int u^2 \, du = \dfrac{u^3}{3} + C = \dfrac{\sec^3 x}{3} + C$

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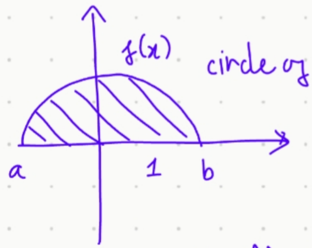
Hence ... $= \frac{u^3}{3} + C = \frac{\sec^3 x}{\ldots}$

§7.3. Trigonometric Substitution.



circle of radius 1 → want to find area

$$\Rightarrow \text{Area} = \int_a^b f(x)\,dx$$

⇒ Need to know what is $f(x)$?

The circle has equation: $x^2 + y^2 = 1$.

1) $\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases} \Rightarrow \cos^2\theta + \sin^2\theta = 1.$

2) $x^2 + y^2 = 1 \Rightarrow y^2 = 1 - x^2 \Rightarrow \left[\begin{array}{l} y = \sqrt{1-x^2} \\ y = -\sqrt{1-x^2} \end{array}\right.$

so if we want to compute the area of upper half disk

→ we put $f(x) = \sqrt{1-x^2}$

$\Rightarrow A = \int_{-1}^{1} \sqrt{1-x^2}\,dx.$ $\Rightarrow \sqrt{a^2 - x^2}$, $a = 1$

let try substitution $u = x^2 \Rightarrow du = 2x\,dx$ ⇒ not possible

⇒ Can not use substitution.

⇒ We can use trigonometric substitution.

Put $x = 1\sin\theta \Rightarrow dx = \cos\theta\,d\theta$

$\begin{cases} x = 1 \\ x = -1 \end{cases} \Rightarrow \begin{cases} \theta = \frac{\pi}{2} \\ \theta = -\frac{\pi}{2} \end{cases}$ $\sin\frac{\pi}{2} = 1$, $\sin(-\frac{\pi}{2}) = -1$

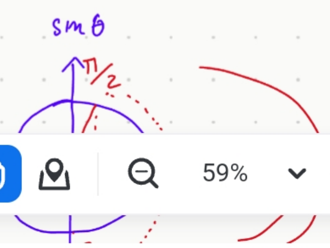
Put $x = \sin x$ or $x = \cos x$

$$\Rightarrow \text{Area} = \int_{-1}^{1} \sqrt{1-x^2}\,dx$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta \qquad \sin^2\theta + \cos^2\theta = 1.$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{\cos^2\theta}\,\cos\theta\,d\theta$$

$$= \int^{\pi/2} \ldots$$

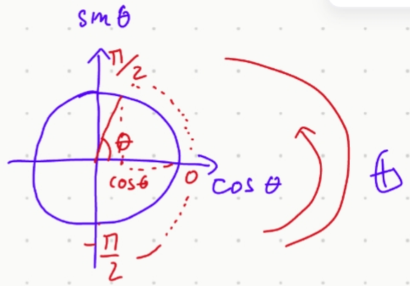


Since $\cos\theta > 0$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$$= \int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta \qquad \sin^2\theta+\cos^2\theta=1.$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{\cos^2\theta}\,\cos\theta\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} |\cos\theta|\,\cos\theta\,d\theta$$



Since $\cos\theta \geq 0$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$|\cos\theta| = \cos\theta$

$$\Rightarrow \text{Area} = \int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]\Bigg|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \underbrace{\frac{\sin\pi}{2}}_{0}\right) - \left(-\frac{\pi}{2} + \underbrace{\frac{\sin(-\pi)}{2}}_{0}\right)\right]$$

$$= \frac{\pi}{2}.$$

If $\cos x$ or $\sin x$ is alone inside the integral with high power

$\Rightarrow$ reduce the power

$\cos 2\theta = 2\cos^2\theta - 1.$

$\Rightarrow \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

General: $a > 0$

1) $\int \sqrt{a^2-x^2}\,dx$: In the example above $a = 1$.

we

Put $x = a\sin\theta$

$\Rightarrow$ $dx = a\cos\theta\,d\theta$

$$= \int \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\,d\theta$$

Use relation $\sin^2\theta + \cos^2\theta = 1$, have

$$= \int |a\cos\theta|\; a\cos\theta\,d\theta$$

$$= a^2 \int |\cos\theta|\,\cos\theta\,d\theta$$

2) Table of trigonometric substitutions.

a) $\sqrt{a^2-x^2}$: $x = a\sin\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$1 - \sin^2\theta = \cos^2\theta.$

$\sqrt{a^2+x^2}$:

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$\dots d\theta$

2) Table of trigonometric substitutions.

a) $\sqrt{a^2-x^2}$: $x = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$1-\sin^2\theta = \cos^2\theta$.

b) $\sqrt{a^2+x^2}$: $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$1+\tan^2\theta = \sec^2\theta$

c) $\sqrt{x^2-a^2}$: $x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$

$\sec^2\theta - 1 = \tan^2\theta$.

Example.

1) $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$ $\quad\sqrt{a^2-x^2}$ with $a=3$

Put $x = 3\sin\theta \Rightarrow dx = 3\cos\theta\, d\theta$ $\quad -\frac{\pi}{2} \le \theta \le$

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{\sqrt{3^2-3\sin^2\theta}}{9\sin^2\theta}\, 3\cos d\theta$$

$$= \int \frac{3\cos\theta \cdot 3\cos\theta}{9\sin^2\theta}\,d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \cot^2\theta\, d\theta$$

$$= \int (\csc^2\theta - 1)\,d\theta$$

$$= -\cot\theta - \theta + C.$$

$$= -\frac{\cos\theta}{\sin\theta} - \theta + C$$

$$= -\frac{\sqrt{1-x^2/9}}{\dots}$$

$x = 3\sin\theta \Rightarrow \sin\theta = $

$\Rightarrow \theta = \arcsin\left(\frac{x}{3}\right)$

and $\cos\theta = \sqrt{1-\sin^2\theta}$

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Example.

1) $\int \frac{\sqrt{9-x^2}}{x^2}\, dx \qquad \sqrt{a^2-x^2}$ with $a=3$

Put $x = 3\sin\theta \Rightarrow dx = 3\cos\theta\, d\theta \qquad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\int \frac{\sqrt{9-x^2}}{x^2}\, dx = \int \frac{\sqrt{3^2 - 3\sin^2\theta}}{9\sin^2\theta}\, 3\cos d\theta$$

$$= \int \frac{3\cos\theta \cdot 3\cos\theta}{9\sin^2\theta}\, d\theta$$

$$= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta$$

$$= \int \cot^2\theta\, d\theta$$

$$= \int (\csc^2\theta - 1)\, d\theta$$

$$= -\cot\theta - \theta + C.$$

$$= -\frac{\cos\theta}{\sin\theta} - \theta + C$$

$$= -\frac{\sqrt{1 - x^2/9}}{x/3} - \arcsin\left(\frac{x}{3}\right) + C.$$

$x = 3\sin\theta \Rightarrow \sin\theta = \frac{x}{3}$

$\Rightarrow \theta = \arcsin\left(\frac{x}{3}\right)$

and $\cos\theta = \sqrt{1-\sin^2\theta}$

$= \sqrt{1 - \frac{x^2}{9}}$

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